# MATH 579 Exam 1 Solutions

1. You throw 20 darts, that all hit a dartboard in the shape of a square of side length 1. You want to draw a bullseye in the shape of a square of side length 1/4, that contains at least two of your darts. Prove that you can always do this.

   Divide the dartboard as a $4 \times 4$ checkerboard. If any dart should land on an edge exactly, it may be placed arbitrarily into any adjacent small square. You threw 20 darts, which land into 16 squares, and hence by the PHP at least one small square (of side length 1/4) must contain at least two darts.

2. Your nemesis chooses a positive integer $n$. Prove that there are positive integers $a > b$ such that 10 divides $n^a - n^b$.

   Consider the units digits of each of $n^1, n^2, \ldots, n^{11}$. There are eleven of these, chosen from $\{0, 1, \ldots, 9\}$, hence by the PHP there are some $a > b$ among $\{1, 2, \ldots, 11\}$ such that $n^a$, $n^b$ have the same units digit $d$. There are therefore integers $x, y$ with $n^a = 10x+d, n^b = 10y+d$, and hence $n^a - n^b = 10x - 10y = 10(x - y)$ is a multiple of 10.

3. Your nemesis chooses seven distinct integers. Prove that some pair has either sum or difference (or both) a multiple of 11.

   Consider the remainders of the seven integers upon division by 11. We map each of them in the natural way to one of the following six sets: $\{0\}, \{1, 10\}, \{2, 9\}, \{3, 8\}, \{4, 7\}, \{5, 6\}$. By PHP there must be some two integers, say $m, n$, whose remainders land in the same set. If they have the same remainder, then $m - n$ is a multiple of 11. If they have different remainders, then $m + n$ is a multiple of 11, because the five doubleton sets were carefully constructed to have this property.

4. (5-10 points) Your nemesis colors each point in the plane red, blue, or green. Prove that there is some rectangle with all four of its corners of the same color.

   We draw a grid consisting of 4 horizontal lines and 82 vertical lines. Each vertical line has four intersections (with the horizontal lines), each of which is one of three colors. There are $3^4 = 81$ possible patterns for these four intersections, so by PHP since $82 > 81$ there are two vertical lines with the same pattern. That pattern has 4 colors, chosen from 3, so by PHP again there is some color appearing at least twice on this pattern (appearing on at least two vertical lines). Those four points form the corners of the desired rectangle.

5. (5-12 points) Your nemesis chooses a sequence of 25 (not necessarily distinct) elements from $\mathbb{Z}_5 \oplus \mathbb{Z}_5$. Prove that some nonempty subsequence sums to $(0, 0)$.

   Let these elements be $x_1, x_2, \ldots, x_{25}$. Consider $x_1, x_1+x_2, x_1+x_2+x_3, \ldots, x_1+x_2+\cdots+x_{25}$. These are 25 sums, each an element of $\mathbb{Z}_5 \oplus \mathbb{Z}_5$. If any are $(0, 0)$ we are done; otherwise, there are only 24 other values in $\mathbb{Z}_5 \oplus \mathbb{Z}_5$, so by PHP two of the sums must agree, say $x_1 + \cdots + x_m = x_1 + \cdots + x_n$ for some $m < n$. Because $\mathbb{Z}_5 \oplus \mathbb{Z}_5$ is a group, $-x_1$ exists, and we may add it to both sides, cancelling the $x_1$ term. Repeating with $x_2, \ldots x_m$, we get $(0, 0) = x_{m+1} + \cdots + x_n$, as desired.

   With care, this argument can be improved, lowering 25 to 9. This is called the Davenport constant of $\mathbb{Z}_5 \oplus \mathbb{Z}_5$, which is known for some finite abelian groups but not others, and is one of the subjects of your instructor's research.